Lecture 24: Large Sets fool Large Linear Tests

Intuition

- Let $A \subseteq \{0,1\}^n$ be a set of size 2^{n-t}
- We want to claim that the uniform distribution over the set A fools (most) large linear tests
- For example, consider A to be the set of n-bit strings that start with t 0s
- Consider any linear test S such that the support of S is restricted only to the first t indices. Then, the output of this linear test is completely biased (it always outputs 0)
- On the other hand, if S has support that is larger than t, then the output of the linear test is uniformly random bit. That is, the uniform distribution over A fools this linear test
- In general, we cannot expect to fool all large support linear tests. For example, we consider A to be the n-bit strings with even number of 1s. The uniform distribution over A does not fool the linear test corresponding to $S = N - 1$

Large Sets fool Large Linear Tests I

- Let $A \subseteq \{0,1\}^n$ such that $|A| = 2^{n-t}$
- Let $\mathbf{1}_{\{A\}}$ be the indicator variable for the subset A
- Note that the uniform distribution over A is represented by the function

$$\frac{1}{|A|}\mathbf{1}_{\{A\}}$$

- Note that the bias of the output of the linear test S is

$$\mathrm{Bias}_{\mathbb{A}}(S) := \frac{N}{|A|}\widehat{\mathbf{1}_{\{A\}}}(S)$$

- Let us evaluate the sum of all the biases corresponding to linear tests S such that $|S| = k$

$$\sum_{S\in\{0,1\}^n : |S|=k} \mathrm{Bias}_{\mathbb{A}}(S)^2 = \left(\frac{N}{|A|}\right)^2 \sum_{S\in\{0,1\}^n : |S|=k} \widehat{\mathbf{1}_{\{A\}}}(S)^2$$

Large Sets fool Large Linear Tests II

- Recall that the KKL Lemma states that, for any $\delta \in (0,1)$ and $f\colon \{0,1\}^n \to \{+1,0,-1\}$, we have

$$\sum_{S\in\{0,1\}^n} \delta^{|S|}\widehat{f}(S)^2 \leqslant \mathbb{P}\left[f(x)\neq 0\colon x \xleftarrow{\$} \{0,1\}^n\right]^{2/1+\delta}$$

- Note that, we have

$$LHS \geqslant \sum_{S\in\{0,1\}^n\colon |S|=k} \delta^k \widehat{f}(S)^2$$

- So, we conclude that

$$\sum_{S\in\{0,1\}^n\colon |S|=k} \widehat{f}(S)^2 \leqslant \frac{1}{\delta^k}\mathbb{P}\left[f(x)\neq 0\colon x \xleftarrow{\$} \{0,1\}^n\right]^{2/1+\delta}$$

- Substituting $f = \mathbf{1}_{\{A\}}$, we get

$$\sum_{S \in \{0,1\}^n : |S| = k} \mathrm{Bias}_{\mathbb{A}}(S)^2 \leqslant \left(\frac{N}{|A|}\right)^2 \cdot \frac{1}{\delta^k} \cdot \left(\frac{|A|}{N}\right)^{2/1+\delta}$$

$$= \frac{1}{\delta^k} \cdot \left(\frac{N}{|A|}\right)^{2\delta/1+\delta}$$

$$\leqslant \frac{1}{\delta^k} \left(\frac{N}{|A|}\right)^{2\delta} = 2^{2t\delta - k \lg \mathrm{e} \ln \delta}$$

- Now, we choose δ that minimizes the RHS above. That value of δ is $\delta = k \lg e / 2t$
- Substituting this value of δ we get

$$\sum_{S \in \{0,1\}^n : |S| = k} \mathrm{Bias}_{\mathbb{A}}(S)^2 \leqslant \left(\frac{2\mathrm{e}t}{k \lg \mathrm{e}}\right)^k$$

- The average bias is

$$\binom{n}{k}^{-1} \sum_{S \in \{0,1\}^n : |S| = k} \mathrm{Bias}_{\mathbb{A}}(S)^2 \leqslant \left(\frac{2\mathrm{e}}{\lg \mathrm{e}} \cdot \frac{t}{n} \right)^k = \Big(O\left(t/n \right) \Big)^k$$

Tightness of the Bound

- The bound we obtain above is essentially tight
- Consider A such that the first t bits of its elements are all 0
- Note that $\binom{t}{k}$ linear tests have bias 1
- The remaining linear tests have bias 0
- So, the average bias is

$$\binom{t}{k}\binom{n}{k}^{-1} \geqslant \left(\frac{1}{\mathrm{e}} \cdot \frac{t}{n}\right)^k$$